

Hitchin's equations and integrability of BPS Z_N strings in Yang-Mills theories

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ABSTRACT: We show that Z_N string's BPS equations are equivalent to the Hitchin's equations (or self-duality equation) and also to the zero curvature condition. We construct the general form for BPS Z_N string solutions for arbitrary simple gauge groups with non-trivial center. Depending on the vacuum solutions considered, the Z_N string's BPS equations reduce to different two dimensional integrable field equations. For a particular vacuum we obtain the equation of affine Toda field theory.

KEYWORDS: Integrable Equations in Physics, Integrable Hierarchies.

Contents

1.	Introduction	1
2.	BPS Z_N strings equations and the Hitchin's equations	2
3.	BPS Z_N string solutions	6
4.	A vacuum solution and affine Toda field theories	9
5.	Rotationally symmetric solutions	11

1. Introduction

In $SU(N)$ QCD, it is believed that particle confinement in the strong coupling regime happens due to chromoelectric strings (QCD strings). Many properties of the QCD strings have been studied intensely in the last years using lattice calculation. On the other hand, it is believed that QCD strings in the strong coupling may be dual to chromomagnetic strings in the Higgs phase in weak coupling, which are easier to study analytically. Since QCD strings in confining phase should be formed only by $SU(3)$ gauge fields and not $U(1)$ gauge fields, in recent years we are analyzing some properties of chromomagnetic Z_N strings solutions which appear in a theory with non-Abelian simple gauge group G (without $U(1)$ factors) broken to its center.

The Z_N string solutions have many features similar to the QCD strings. In particular they are associated to coweights of representations of G (or equivalently to weights of the dual group¹ G^\vee) and their topological sectors are associated to the center elements of the gauge group G . More precisely, the coweights of G can be separated in cosets associated to nodes of the extended Dynkin diagram of G . All Z_N string solutions associated to coweights in a given coset belong to the same topological sector [1]. The Z_N strings associated to the fundamental weights of different representations can have different tensions and for different vacuum solutions, the BPS bounds for the tensions can satisfy either the sine law scaling or the Casimir scaling [1, 2], differently from the non-Abelian semi-local BPS string solutions with gauge group $SU(N) \times U(1)$ where the tension is only due magnetic flux in the $U(1)$ direction and it depends on the $U(1)$ winding number [3]. It is important to note that the Casimir scaling and the sine law scaling considered in [1, 2] are lower bounds for the non-BPS Z_N string tensions.

Previously we analyzed the Z_N string in soft broken $\mathcal{N} = 2$ [4, 5] and $\mathcal{N} = 4$ [1] Super Yang-Mills theories, but in [2] and here we do not constraint the potential to be

¹We shall consider the dual group G^\vee as the covering group associated to the dual algebra \mathfrak{g}^\vee .

supersymmetric since we are interested in studying some general properties at the classical level of the Z_N strings which may be useful for QCD and not necessarily confinement in supersymmetric theories. The Z_N strings does not necessarily point in a direction in the Cartan subalgebra (CSA). However, since the monopoles' magnetic flux is in the direction of the CSA [6], we only consider Z_N string solutions with flux in the CSA which are the relevant for confinement of these monopoles [5, 1] which may be dual to particle confinement. This result is analogous to the Abelian dominance observed in QCD.

In the present work we show that Z_N string's BPS equations [4, 1] for Yang-Mills theories with scalars in the adjoint are equivalent to the Hitchin's equations [7] and consequently to the four dimensional self-duality equation [7] and also equivalent to the zero curvature condition [11], implying that this set of BPS Z_N string solutions in non-Abelian Yang-Mills theories is (quasi-)integrable. Integrability of BPS vortices in Abelian-Higgs theory was recently considered in [8]. Integrability of other soliton solutions for theories in dimensions higher than two are analyzed in [9]. In recent years, integrability also had a renewed interest in gauge and string theories [10]. On the other hand, Hitchin's equations appeared in many distinct problems as for example in Matrix string theory [11, 12] and more recently in connection with the geometric Langlands program [13]. Integrability of Hitchin's equations and self-duality equations are also discussed in [14, 15].

The equivalence of BPS Z_N string equations with Hitchin's equations, self-duality equations and zero curvature condition is interesting because allows us to apply methods and results of these systems to Z_N string solutions and vice-versa.

In the U(1) Abelian-Higgs theory, the fields of rotationally symmetric BPS string solutions can be written as functions of a field which satisfies Liouville's equation plus a constant and a singularity at the origin [16, 17]. In this work we generalize this result for the Z_N strings showing that, for a particular vacuum responsible for the gauge symmetry breaking, the fields of the rotationally symmetric BPS $Z(N)$ string solutions are functions of a field which satisfies affine Toda field equation with a singularity at the origin.

In this paper we introduce, in section 2, some general results for BPS Z_N strings and show the equivalence of the Z_N string BPS conditions with the Hitchin's equations, and consequently to self-duality equation and the zero curvature condition. In section 3 we construct an Ansatz for the Z_N strings and show that the Z_N string's BPS equations reduce to two dimensional integrable theories equations. In section 4 we show that for a particular vacuum, the BPS Z_N string solutions reduces to the equation of affine Toda theory which is a deformation of conformal Toda theory. In section 5 we analyze the special case of rotationally symmetric solutions. These solutions resemble the Riemannian or stringy instantons of Matrix string theories [11, 12].

2. BPS Z_N strings equations and the Hitchin's equations

Let us consider Yang-Mills-Higgs theories with arbitrary gauge group G which is simple, connected and simply connected. In order to exist strings and confined monopoles we shall consider theories with two complex scalars fields ϕ_s , $s = 1, 2$, in the adjoint representation of G . We also consider that the vacuum solutions ϕ_1^{vac} , ϕ_2^{vac} produce the symmetry breaking

pattern

$$G \xrightarrow{\phi_1^{\text{vac}}} \text{U}(1)^r \xrightarrow{\phi_2^{\text{vac}}} C_G, \quad (2.1)$$

where r is the rank of G and C_G its center, which we consider to be nontrivial. The Lagrangian of the theory we are to study is

$$\mathcal{L} = -\frac{1}{4}G_{a\mu\nu}G_a^{\mu\nu} + \sum_{s=1}^2 \frac{1}{2} (D_\mu\phi_s)_a^* (D^\mu\phi_s)_a - V(\phi, \phi^*) \quad (2.2)$$

where a is a Lie algebra index, $D_\mu = \partial_\mu + ie[W_\mu, \]$ and $G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + ie[W_\mu, W_\nu]$. Let

$$\begin{aligned} z &= x^1 + ix^2, \\ \partial_z &= \frac{1}{2}(\partial_1 - i\partial_2), \\ W_z &= \frac{1}{2}(W_1 - iW_2), \end{aligned} \quad (2.3)$$

and $B_i = -\epsilon_{ijk}G_{jk}/2$. For a static string solution with cylindrical symmetry in the x_3 direction, the string BPS equations for a theory with gauge group G without $\text{U}(1)$ factors are [4, 1]²

$$B_{3a} = -d_a, \quad (2.4)$$

$$D_z\phi_s = 0, \quad (2.5)$$

$$D_{\bar{z}}\phi_s^\dagger = 0, \quad (2.6)$$

$$V(\phi, \phi^*) - \frac{1}{2}(d_a)^2 = 0, \quad (2.7)$$

with

$$d = \frac{e}{2} \sum_{s=1}^2 [\phi_s^\dagger, \phi_s] - \frac{em}{2} \text{Re}(\phi_1),$$

where m is a non-negative mass parameter. The string tension satisfies the bound

$$T \geq \frac{me}{2} |\phi_1^{\text{vac}}| |\Phi_{\text{st}}| \quad (2.8)$$

where

$$\Phi_{\text{st}} = \frac{1}{|\phi_1^{\text{vac}}|} \int d^2x \text{Tr} [\text{Re}(\phi_1) B_3] \quad (2.9)$$

is the string flux in the ϕ_1 direction, with the integral being taking in the plane orthogonal to the string. The equality in eq. (2.8) happens only for the strings satisfying the BPS equations. We shall consider

$$V(\phi, \phi^*) = \frac{1}{2}(d_a)^2, \quad (2.10)$$

which guarantee that equation (2.7) is automatically fulfilled. Note that the BPS equation (2.7) does not restrict the potential to have this form. In [4, 5, 1] we considered soft

²For the sake of simplicity we shall only consider the string solutions which have positive flux Φ_{st} . For the antistrings, one must use opposite signs in some of these equations as discussed in our previous works.

broken $\mathcal{N} = 2$ and $\mathcal{N} = 4$ potentials. Similarly to the Prasad-Sommerfield limit [18] for BPS monopoles, we take the limit $m \rightarrow 0$ [4] in order for the BPS string equations to be consistent with the equations of motion and retain the potential terms responsible for the breaking of G into its center C_G . Note that in this limit, the tensions $T \rightarrow 0$, although the ratio of the string tensions are finite and can satisfy the Casimir scaling law or the sine law as discussed in [2] and in section 4.

Let g be the Lie algebra associated to the gauge group G and let the generators $H_i, i = 1, 2, \dots, r$, form a basis for the Cartan subalgebra (CSA) h with rank r . Let us adopt the Cartan-Weyl basis in which

$$\begin{aligned} \text{Tr}(H_i H_j) &= \delta_{ij}, \\ \text{Tr}(E_\alpha E_\beta) &= \frac{2}{\alpha^2} \delta_{\alpha+\beta}, \end{aligned}$$

where the trace is taken in the adjoint representation. In this basis, the commutation relations read

$$\begin{aligned} [H_i, E_\alpha] &= (\alpha)^i E_\alpha, \\ [E_\alpha, E_{-\alpha}] &= \frac{2\alpha}{\alpha^2} \cdot H, \end{aligned} \tag{2.11}$$

where α are roots and the upper index in $(\alpha)^i$ means the component i of α . We denote by α_i and $\lambda_i, i = 1, 2, \dots, r$, the simple roots and fundamental weights of g respectively and

$$\alpha_i^\vee = \frac{2\alpha_i}{\alpha_i^2}, \quad \lambda_i^\vee = \frac{2\lambda_i}{\alpha_i^2} \tag{2.12}$$

are the simple co-roots and fundamental co-weights of g , and are also the simple roots and fundamental weights of the dual algebra g^\vee . They satisfy the relations

$$\alpha_i \cdot \lambda_j^\vee = \alpha_i^\vee \cdot \lambda_j = \delta_{ij}.$$

Moreover,

$$\alpha_i = K_{ij} \lambda_j \tag{2.13}$$

where

$$K_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j^2} \tag{2.14}$$

is the Cartan matrix associated to g . We denote by ψ the highest root of g . Considering the convention that $\psi^2 = 2$, the highest root can be written as

$$\psi = \sum_{i=1}^r m_i \alpha_i^\vee \tag{2.15}$$

where m_i are integers which are the levels (or marks) of the fundamental representations which have λ_i as highest weights. For $SU(n)$, $m_i = 1$ for $i = 1, 2, \dots, n - 1$.

In order to produce the symmetry breaking (2.1) we can consider a general vacuum solution

$$\phi_1^{\text{vac}} = v \cdot H, \quad v = v_i \lambda_i^\vee, \quad (2.16)$$

$$\phi_2^{\text{vac}} = \sum_{l=0}^r b_l E_{-\alpha_l}, \quad (2.17)$$

where $\alpha_0 = -\psi$ is the negative of the highest weight, v_i are non-vanishing real constants, b_l are real constants and for $l = 1, 2, \dots, r$ they can not vanish in order to G to be broken into C_G . Comparing with the general vacuum solutions considered in [1, 2], in (2.17) we add a term associated to the root $-\alpha_0$ which does not change the symmetry breaking but change some properties of the vacuum solutions as we shall explain bellow.

We usually consider Z_N strings solutions with the gauge fields in the CSA with H_i , $i = 1, 2, \dots, r$ as basis generators which are the relevant for confinement of the standard monopole solution, since the monopoles have magnetic flux in direction of the CSA. Then, as discussed in our previous works [4, 1], since the gauge fields are everywhere in the Cartan subalgebra, from the BPS equations $D_z \phi_1 = 0$ and $D_{\bar{z}} \phi_1^\dagger = 0$ results that the field $\phi_1(x)$ is constant and equal to its asymptotic form (2.16), i.e.,

$$\phi_1(x) = v \cdot H. \quad (2.18)$$

Then, the BPS equations (2.4)–(2.6) in the limit $m \rightarrow 0$ reduce to

$$\begin{aligned} G_{\bar{z}z} &\rightarrow -\frac{ie}{4} [\phi_2^\dagger, \phi_2], \\ D_z \phi_2 &= 0, \\ D_{\bar{z}} \phi_2^\dagger &= 0, \end{aligned} \quad (2.19)$$

which are exactly the Hitchin's equations [7].

As it is known, these equations are equal to a reduction to two dimensions of the self-duality equation in Euclidean four dimensions [7],

$$\mathcal{G}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \mathcal{G}_{\rho\sigma},$$

with gauge fields $\mathcal{W}_i = W_i$ for $i = 1, 2$ and

$$\begin{aligned} \mathcal{W}_3 &= \phi_{2r}, \\ \mathcal{W}_4 &= \phi_{2i}, \end{aligned} \quad (2.20)$$

where ϕ_{2r} and ϕ_{2i} are respectively the real and imaginary parts of ϕ_2 and imposing that the fields does not depend on the extra dimensions with coordinates x^3 and x^4 .

The Equations (2.19) can also be written in the form of a zero curvature condition in two dimensions considering the connection [11]

$$\begin{aligned} \mathcal{A}_z &= W_z + \frac{\lambda}{2} \phi_2^\dagger, \\ \mathcal{A}_{\bar{z}} &= W_{\bar{z}} - \frac{1}{2\lambda} \phi_2, \end{aligned} \quad (2.21)$$

where λ is a spectral parameter. Then,

$$\begin{aligned} \mathcal{F}_{\bar{z}z} &= \partial_{\bar{z}}\mathcal{A}_z - \partial_z\mathcal{A}_{\bar{z}} + ie[\mathcal{A}_{\bar{z}}, \mathcal{A}_z] \\ &= \left(G_{\bar{z}z} + \frac{ie}{4} [\phi_2^\dagger, \phi_2] \right) + \frac{\lambda}{2} D_{\bar{z}}\phi_2^\dagger + \frac{1}{2\lambda} D_z\phi_2. \end{aligned}$$

Therefore, the system of equations (2.19) is equivalent to a zero curvature condition which implies the classical integrability of the set of BPS Z_N string solutions. More precisely, due to the limit $m \rightarrow 0$, we have the condition $\mathcal{F}_{\bar{z}z} \rightarrow 0$, which we could call quasi-integrability condition. However, for simplicity we will use the equal sign in the following integrable equations.

The flat connection $\mathcal{A}_z, \mathcal{A}_{\bar{z}}$ can be written in terms of the self-dual fields \mathcal{W}_i (2.20) as

$$\begin{aligned} \mathcal{A}_z &= \frac{1}{2}(\mathcal{W}_1 - i\mathcal{W}_2) + \frac{\lambda}{2}(\mathcal{W}_3 - i\mathcal{W}_4), \\ \mathcal{A}_{\bar{z}} &= \frac{1}{2}(\mathcal{W}_1 + i\mathcal{W}_2) - \frac{1}{2\lambda}(\mathcal{W}_3 + i\mathcal{W}_4). \end{aligned} \tag{2.22}$$

The equivalence of BPS Z_N string equations, with the Hitchin's equations, self-duality equations and zero curvature condition is interesting because it allows to apply methods and results of these systems to the Z_N string solutions and vice-versa.

In the next section we show that for Z_N string solutions constructed from different vacuum are associated to different integrable field equations.

3. BPS Z_N string solutions

In the Higgs phase of the theory, when G is broken to its center C_G which we consider to be non-trivial, there exist Z_N string solutions and the monopoles are confined by these strings. In order to have finite string tension, asymptotically these solutions have the form

$$\begin{aligned} \phi_s(\varphi, \rho \rightarrow \infty) &= g(\varphi)\phi_s^{\text{vac}}g(\varphi)^{-1}, \quad s = 1, 2, \\ W_i(\varphi, \rho \rightarrow \infty) &= \frac{i}{e}(\partial_i g(\varphi))g(\varphi)^{-1}, \quad i = 1, 2, \end{aligned} \tag{3.1}$$

where ϕ_s^{vac} are the vacuum solutions (2.16), (2.17), ρ and φ are the radial and angular coordinates. In order for the configuration to be single valued, $g(\varphi + 2\pi)g(\varphi)^{-1} \in C_G$. Considering

$$g(\varphi) = \exp i\varphi M, \quad \text{where } M = \omega \cdot H$$

it implies that $\exp(2\pi i\omega \cdot H) \in C_G$, which results that $\omega \in \Lambda_w(G^\vee)$, where

$$\Lambda_w(G^\vee) = \left\{ \omega = \sum_{i=1}^r n_i \lambda_i^\vee, \quad n_i \in \mathbb{Z} \right\} \tag{3.2}$$

is the coweight lattice of G or equivalently the weight lattice of the dual group G^\vee . Then, using the vacuum solutions (2.16), (2.17), the asymptotic form of the Z_N string solution (3.1)

can be written as

$$\begin{aligned}
 \phi_1(\varphi, \rho \rightarrow \infty) &= v \cdot H, \\
 \phi_2(\varphi, \rho \rightarrow \infty) &= \sum_{i=0}^r b_i \{ \exp(-i\varphi\omega \cdot \alpha_i) \} E_{-\alpha_i}, \\
 W_i(\varphi, \rho \rightarrow \infty) &= \frac{\epsilon_{ij} x^j}{e\rho^2} \omega \cdot H, \quad i = 1, 2.
 \end{aligned}
 \tag{3.3}$$

Therefore, for each weight ω of the dual group G^\vee we can construct a string solution. In [1] is shown how these strings are separated in different topological sectors.

As mentioned before, we can take $\phi_1(\varphi, \rho) = v \cdot H$ for the whole space. In order to determine the other fields for the whole space we consider the Ansatz

$$\phi_2(\rho, \varphi) = \sum_{i=0}^r f_i(\rho, \varphi) b_i E_{-\alpha_i} \exp(-iY(\rho, \varphi) \cdot \alpha_i).
 \tag{3.4}$$

Similarly to the string solution in the Abelian Higgs model, if ω is such that for a given α_i , the scalar product $\omega \cdot \alpha_i \neq 0$, then the corresponding function $f_i(\rho, \varphi)$ must have some zeros since from the asymptotic form (3.3) we see that the terms with $\omega \cdot \alpha_i \neq 0$ have non-vanishing winding number. We can rewrite this Ansatz, similarly to the string solutions in the Abelian Higgs model [16, 19], as

$$\phi_2(\rho, \varphi) = G(\rho, \varphi) \phi_2^{\text{vac}} G^{-1}(\rho, \varphi)
 \tag{3.5}$$

where ϕ_2^{vac} is the vacuum solution (2.17) and

$$G(\rho, \varphi) = \exp[Z(\rho, \varphi) \cdot H], \quad Z(\rho, \varphi) = -\frac{e}{2} X(\rho, \varphi) + iY(\rho, \varphi),$$

with $Z(\rho, \varphi)$, $X(\rho, \varphi)$ and $Y(\rho, \varphi)$ being r component real functions with $2 \ln f_i = eX \cdot \alpha_i$. The points where $f_i(\rho, \varphi)$ vanishes, $X \cdot \alpha_i$ has a logarithmic singularity. From the asymptotic form (3.3) we can conclude that for $\rho \rightarrow \infty$,

$$\begin{aligned}
 X(\rho \rightarrow \infty, \varphi) &= 0, \\
 Y(\rho \rightarrow \infty, \varphi) &= \varphi\omega.
 \end{aligned}$$

For the special case of rotationally symmetric solutions, we can consider $Y(\rho, \varphi) = \varphi\omega$ and $X(\rho, \varphi) = X(\rho)$ is a radial function. This ansatz can be used for any Z_N string solution, not only BPS.

From eq. (3.5) results that

$$\partial_z \phi_2 = [(\partial_z G) G^{-1}, \phi_2].$$

Therefore, from the BPS equation $D_z \phi_2 = 0$, we can conclude that

$$W_z = \frac{i}{e} (\partial_z G) G^{-1} + F_z = \frac{i}{e} \partial_z (Z \cdot H) + F_z
 \tag{3.6}$$

where $F_z(x)$ is a Lie algebra valued function which commutes with ϕ_2 . On the other hand, from the BPS equation $D_z\phi_1 = 0$ and the fact that $\phi_1(x) = v \cdot H$, implies $F_z(x)$ should belong to the CSA. Since ϕ_2 is not in the CSA, it implies that $F_z(x) = 0$. Similarly, by computing $\partial_{\bar{z}}\phi_2^\dagger$ and repeating the above argument we can conclude that

$$W_{\bar{z}} = \frac{i}{e}\partial_{\bar{z}}(G^{-1})^\dagger G^\dagger = -\frac{i}{e}\partial_{\bar{z}}(Z^\dagger \cdot H). \quad (3.7)$$

Note that $G^\dagger \neq G^{-1}$.

Therefore, from the first BPS equation in (2.19) results that X satisfies³

$$\partial_{\bar{z}}\partial_z(X \cdot H) - \frac{e}{4}\left[e^{eX \cdot H}(\phi_2^{\text{vac}})^\dagger e^{-eX \cdot H}, \phi_2^{\text{vac}}\right] = 0. \quad (3.8)$$

This is the equation of motion of an *Euclidean* two dimensional integrable system since it equivalent to the zero curvature condition with the connection (2.21) using the fields configurations (3.5), (3.6) and (3.7) (which are solutions of $D_z\phi_2 = 0$ and $D_{\bar{z}}\phi_2^\dagger = 0$), that is

$$\begin{aligned} \mathcal{A}_z &= \frac{i}{e}\partial_z(Z \cdot H) + \frac{\lambda}{2}\exp(-Z^\dagger \cdot H)(\phi_2^{\text{vac}})^\dagger \exp(Z^\dagger \cdot H), \\ \mathcal{A}_{\bar{z}} &= -\frac{i}{e}\partial_{\bar{z}}(Z^\dagger \cdot H) - \frac{1}{2\lambda}\exp(Z \cdot H)\phi_2^{\text{vac}} \exp(-Z \cdot H). \end{aligned} \quad (3.9)$$

Using the fact that $\phi_2^{\text{vac}} = \sum_{l=0}^r b_l E_{-\alpha_l}$, we can write eq. (3.8) as

$$\partial_{\bar{z}}\partial_z X - \frac{e}{4}\sum_{j=0}^r b_j^2 \alpha_j^\vee e^{e\alpha_j \cdot X} = 0, \quad (3.10)$$

remembering that X is an r component scalar field. For the vacuum solutions with $b_0 = 0$, we define

$$X_{\alpha_i} = \alpha_i \cdot X, \quad i = 1, 2, \dots, r.$$

Then, eq. (3.10) can be written as

$$\partial_{\bar{z}}\partial_z X_{\alpha_i} - \frac{e}{4}\sum_{j=1}^r K_{ij} b_j^2 \exp(eX_{\alpha_j}) = 0, \quad (3.11)$$

where K_{ij} is the Cartan matrix (2.14).

On the other hand, for vacuum solutions with $b_0 \neq 0$, we define

$$X_{\alpha_i} = \alpha_i \cdot X, \quad i = 0, 1, 2, \dots, r. \quad (3.12)$$

In this case, eq. (3.10) can be written as

$$\partial_{\bar{z}}\partial_z X_{\alpha_i} - \frac{e}{4}\sum_{j=0}^r \widehat{K}_{ij} b_j^2 \exp(eX_{\alpha_j}) = 0, \quad i = 0, 1, 2, \dots, r \quad (3.13)$$

³In order to arrive to this equation we are not considering the points where X has a singularity. We discuss more on this issue in the last section.

where \widehat{K}_{ij} is the extended Cartan matrix which has the same form as the Cartan matrix eq. (2.14), but with $i, j = 0, 1, \dots, r$, and are associated to untwisted affine Lie algebras. However, from (2.15) and the fact that $\alpha_0 = -\psi$ we can conclude that

$$\sum_{i=0}^r m_i \alpha_i^\vee = 0,$$

where we consider $m_0 = 1$. Therefore, the fields X_{α_i} in (3.12) are not independent but satisfy the constraint

$$\sum_{i=0}^r \frac{2m_i}{\alpha_i^2} X_{\alpha_i} = 0.$$

Therefore, equation (3.13) must be subject to this constraint [20].

As we mentioned before, for a Z_N string solution associated to a vector ω , for the terms in eq. (3.4) where $\omega \cdot \alpha_i \neq 0$, the corresponding function f_i must have some zeros and hence $X_{\alpha_i} = \alpha_i \cdot X$ has logarithmic singularities. Therefore equations (3.11) and (3.13) are valid except at the singularities of X_{α_i} . Similarly to the Abelian case [16, 17], we can allow for these singularities by including delta-functions on the right hand side of the above equations.

4. A vacuum solution and affine Toda field theories

Let us now consider a concrete vacuum solution. In order to be a vacuum solution of the potential (2.10), the constants in (2.16), (2.17), must satisfy the relation

$$m (K^{-1})_{ij} v_j = b_i^2 - m_i b_0^2. \tag{4.1}$$

In [1, 2] we analyzed two vacuum solutions with $b_0 = 0$, which are valid for any gauge group G :

a) The first vacuum solution we considered was

$$v_i = a, \tag{4.2}$$

$$b_i = \sqrt{ma \sum_{j=1}^r (K^{-1})_{ij}} = \sqrt{ma\delta \cdot \lambda_i}, \quad i = 1, 2, \dots, r, \tag{4.3}$$

where a is a positive real constant and $\delta = \sum_{i=1}^r \lambda_i^\vee$ is the dual Weyl vector. With this vacuum, the Z_N strings tensions satisfy the Casimir scaling [1].

b) The second vacuum solution was

$$v_i = ay_i^{(1)}, \tag{4.4}$$

$$b_i = \frac{1}{2 \sin \frac{\pi}{2h}} \sqrt{may^{(1)}}, \quad i = 1, 2, \dots, r, \tag{4.5}$$

where a is a positive real constant and $y_i^{(1)}$ are the components of the Perron-Frobenius eigenvectors of K_{ij} associated to the eigenvalue $4 \sin^2 \frac{\pi}{2h}$. With this vacuum, the Z_N strings tensions satisfy the sine law scaling [2].

As mentioned before, in order for the Z_N string's BPS equations to be consistent with the equations of motion we must take the limit $m \rightarrow 0$. Therefore, for the constants b_i or equivalently ϕ_2^{vac} to be finite, we must take $a \rightarrow \infty$ (and therefore $v_i \rightarrow \infty$) keeping the product ma finite [2]. Another possibility we use here is to consider the vacuum solutions (2.16), (2.17) with $b_0 \neq 0$ in which case we can keep a finite. One can see this from eq. (4.1) which implies that

$$b_i = \sqrt{m(K^{-1})_{ij}v_j + m_i b_0^2}. \quad (4.6)$$

From this equation, considering that v_j is finite, when we take $m \rightarrow 0$, it implies that

$$b_i \rightarrow \sqrt{m_i} b_0$$

which is finite with b_0 being an arbitrary non-vanishing constant. That result hold when the components v_j satisfy either (4.2) or (4.4). In each case, the ratio of tensions of the BPS Z_N strings will continue to satisfy the Casimir scaling or the sine law scaling respectively. The vacuum solution (2.17) will stay

$$\phi_2^{\text{vac}} = b_0 \sum_{l=0}^r \sqrt{m_l} E_{-\alpha_l} = b_0 E, \quad (4.7)$$

where

$$E = \sum_{i=0}^r \sqrt{m_i} E_{\alpha_i}$$

and therefore asymptotically

$$\phi_2(\varphi, \rho \rightarrow \infty) = b_0 \sum_{i=0}^r \sqrt{m_i} \{ \exp(-i\varphi\omega \cdot \alpha_i) \} E_{-\alpha_i}. \quad (4.8)$$

The generator E satisfy $[E, E^\dagger] = 0$. Hence it is diagonalizable and can be embedded in a new Cartan subalgebra. This generator was originally introduced by Konstant [21].

For this vacuum we can write (3.8) as

$$\partial_{\bar{z}} \partial_z (X \cdot H) - \frac{eb_0^2}{4} \left[e^{eX \cdot H} E^\dagger e^{-eX \cdot H}, E \right] = 0, \quad (4.9)$$

It can also be written as

$$\partial_{\bar{z}} \partial_z X - \frac{eb_0^2}{4} \sum_{j=0}^r m_j \alpha_j^\vee \exp(e\alpha_j \cdot X) = 0 \quad (4.10)$$

or

$$\partial_{\bar{z}} \partial_z X_{\alpha_i} - \frac{eb_0^2}{4} \sum_{j=0}^r \widehat{K}_{ij} m_j \exp(eX_{\alpha_j}) = 0. \quad (4.11)$$

Eq. (4.9) (or (4.10), (4.11)) is the equation of motion of Euclidean two dimensional integrable affine Toda field theory associated to the affine untwisted Lie algebra \widehat{g} obtained from g , with coupling constant e , equal to the coupling constant of the gauge theory,

and mass parameter equal to eb_0 . In [2] was shown that the spectrum of BPS string tensions in the second vacuum solution (4.4) coincide with the solitons mass spectrum of the corresponding affine Toda theory.

It is interesting to note that the monopole's BPS equations with spherical symmetry reduces to the equation of conformal Toda theory [22]. The equation of Affine Toda theory was also obtained from Hitchin's equations for $U(N)$ matrix theory in [11], but with some differences as we discuss in the next section. A relation between Hitchin equations and integrable systems was also consider in [14]. For $g = su(2)$, considering $\alpha_1 = 1 = -\alpha_0$, equation (4.10) reduces to the sinh-Gordon equation

$$\partial_{\bar{z}}\partial_z X - \frac{eb_0^2}{2} \sinh(eX) = 0.$$

5. Rotationally symmetric solutions

Let us now consider the special case of rotationally symmetric solutions. In this case for a Z_N string associated with the vector ω of the weight lattice of dual group G^\vee , $Y(\rho, \varphi) = \varphi\omega$ and $X(\rho, \varphi) = X(\rho)$ is a radial function, and hence

$$Z(\rho, \varphi) = -\frac{e}{2}X(\rho) + i\varphi\omega. \tag{5.1}$$

Therefore, in this case the scalar fields has the form

$$\phi_1(\varphi, \rho) = v \cdot H, \tag{5.2}$$

$$\phi_2(\varphi, \rho) = b_0 \sum_{i=0}^r \sqrt{m_i} \left\{ \exp\left(\frac{e}{2}X \cdot \alpha_i - i\varphi\omega \cdot \alpha_i\right) \right\} E_{-\alpha_i}. \tag{5.3}$$

Generalizing some results from stringy instantons of matrix string theories [11, 12], we can determine the form of the gauge fields: since $2\varphi = -i \ln(z/\bar{z})$, the gauge fields (3.6) and (3.7) are given by

$$\begin{aligned} W_z &= \frac{i}{2} \left[-\partial_z (X \cdot H) + \frac{1}{ez} \omega \cdot H \right], \\ W_{\bar{z}} &= -\frac{i}{2} \left[-\partial_{\bar{z}} (X \cdot H) + \frac{1}{e\bar{z}} \omega \cdot H \right]. \end{aligned} \tag{5.4}$$

Using that $\partial_z(1/\bar{z}) = \pi\delta^{(2)}(x)$, we obtain that the magnetic field of the Z_N string is

$$B_3 = -2iG_{\bar{z}z} = -2 \left[\partial_{\bar{z}}\partial_z (X \cdot H) - \frac{\pi}{e} \omega \cdot H \delta^{(2)}(x) \right]. \tag{5.5}$$

From the requirement of regularity at $z = 0$ implies that near the origin

$$\partial_{\bar{z}}\partial_z (X \cdot H) \sim \frac{\pi}{e} \omega \cdot H \delta^{(2)}(x) + \text{const.}$$

Therefore, near the origin

$$X(\rho \rightarrow 0) \sim -\frac{2}{e} \omega \ln |z| + \text{const.} \tag{5.6}$$

On the other hand, for $\rho \rightarrow \infty$, $X(\rho \rightarrow \infty) \rightarrow 0$, as we mentioned before. Eq. (5.6) is consistent with the fact that in the general case (not necessarily rotationally symmetric), $X_{\alpha_i} = \alpha_i \cdot X$ has a logarithmic singularity if $\alpha_i \cdot \omega \neq 0$.

From (5.2) and (5.5) we conclude that the flux (2.9) of a string associated to the coweight w is

$$\Phi_{\text{st}} = \frac{2\pi \omega \cdot v}{e |v|},$$

which is consistent with the result in [1] using just the asymptotic form of the Z_N string solution (3.3).

From eq. (4.11) and the behavior of the solution near the origin (5.6) we can conclude that for the special case of rotationally symmetric solutions, for a Z_N string associated with a coweight w , the radial function $X(\rho)$ must satisfy

$$\frac{\partial^2 X}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial X}{\partial \rho} - eb_0^2 \sum_{j=0}^r \alpha_j^\vee m_j \exp(e\alpha_j \cdot X) = \frac{4\pi}{e} \omega \delta^{(2)}(x) \quad (5.7)$$

or

$$\frac{\partial^2 X_{\alpha_i}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial X_{\alpha_i}}{\partial \rho} - eb_0^2 \sum_{j=0}^r \widehat{K}_{ij} m_j \exp(eX_{\alpha_j}) = \frac{4\pi}{e} \omega \cdot \alpha_i \delta^{(2)}(x). \quad (5.8)$$

One could arrive directly to this equation with the singularity using (5.1) and (5.4) in the BPS equation (2.4) or in the connection (3.9). That result is similar to the string solution in the Abelian-Higgs theory where for a rotationally symmetric configuration Ansatz the radial function, satisfies a rotationally symmetric form of a Liouville's equation plus a constant with a δ -function at the origin [16, 17]. Note that our condition $m \rightarrow 0$ corresponds to take the limit $a \rightarrow 0$ for the constant appearing in the potential in Abelian-Higgs theory, in which case the radial function of the string solution would satisfy Liouville's equation with a δ -function at the origin.

The Z_N string solutions have great similarity with the Riemannian or stringy instantons of matrix string theories [11, 12], but also some differences: in the matrix stringy theories there is no gauge symmetry breaking. For the stringy instantons, the scalar field has a branch cut in the origin instead of a zero for $\phi_2(\rho, \varphi)$ for the rotationally symmetric Z_N strings. They also have different angular dependence since the Z_N string solutions are associated to classes of $\Pi_1(G/C_G)$. Moreover, the affine Toda equation associated to stringy instantons [11] has a singularity structure different from (5.7).

Eqs. (5.7) or (5.8) with source is equivalent to the homogeneous equation with the boundary (5.6) near the origin. In general to solve "Toda type theories" one can apply Leznov-Saveliev method [23]. Solutions for a similar equation for $g = su(2)$ with singularity were analyzed in [24]. Soliton solutions of Affine Toda theories (which have different boundary condition) were analyzed for example in [25, 20] for different algebras g . Similar methods may be can applied for this case with different boundary conditions.

References

- [1] M.A.C. Kneipp, *Color superconductivity, $Z(N)$ flux tubes and monopole confinement in deformed $N = 2^*$ super Yang-Mills theories*, *Phys. Rev. D* **69** (2004) 045007 [[hep-th/0308086](#)].
- [2] M.A.C. Kneipp, *BPS Z_N string tensions, sine and Casimir laws and integrable field theories*, *Phys. Rev. D* **76** (2007) 125010 [[arXiv:0707.3791](#)].
- [3] T. Vachaspati and A. Achucarro, *Semilocal cosmic strings*, *Phys. Rev. D* **44** (1991) 3067; A. Hanany and D. Tong, *Vortices, instantons and branes*, *JHEP* **07** (2003) 037 [[hep-th/0306150](#)]; R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, *Nonabelian superconductors: vortices and confinement in $N = 2$ SQCD*, *Nucl. Phys. B* **673** (2003) 187 [[hep-th/0307287](#)].
- [4] M.A.C. Kneipp and P. Brockill, *BPS string solutions in non-abelian Yang-Mills theories*, *Phys. Rev. D* **64** (2001) 125012 [[hep-th/0104171](#)].
- [5] M.A.C. Kneipp, *$Z(k)$ string fluxes and monopole confinement in non-Abelian theories*, *Phys. Rev. D* **68** (2003) 045009 [[hep-th/0211049](#)].
- [6] P. Goddard, J. Nuyts and D.I. Olive, *Gauge theories and magnetic charge*, *Nucl. Phys. B* **125** (1977) 1.
- [7] N.J. Hitchin, *The selfduality equations on a Riemann surface*, *Proc. Lond. Math. Soc.* **55** (1987) 59.
- [8] C. Adam, J. Sanchez-Guillen and A. Wereszczynski, *Integrability in theories with local $U(1)$ gauge symmetry*, *J. Phys. A* **40** (2007) 9079 [[hep-th/0702100](#)]; A.D. Popov, *Integrability of vortex equations on Riemann surfaces*, [arXiv:0712.1756](#).
- [9] O. Alvarez, L.A. Ferreira and J. Sanchez Guillen, *A new approach to integrable theories in any dimension*, *Nucl. Phys. B* **529** (1998) 689 [[hep-th/9710147](#)]; H. Aratyn, L.A. Ferreira and A.H. Zimerman, *Toroidal solitons in $3 + 1$ dimensional integrable theories*, *Phys. Lett. B* **456** (1999) 162 [[hep-th/9902141](#)]; *Exact static soliton solutions of $3 + 1$ dimensional integrable theory with nonzero Hopf numbers*, *Phys. Rev. Lett.* **83** (1999) 1723 [[hep-th/9905079](#)]; L.A. Ferreira and J. Sanchez Guillen, *Infinite symmetries in the Skyrme model*, *Phys. Lett. B* **504** (2001) 195 [[hep-th/0010168](#)]; O. Babelon and L.A. Ferreira, *Integrability and conformal symmetry in higher dimensions: a model with exact Hopfion solutions*, *JHEP* **11** (2002) 020 [[hep-th/0210154](#)].
- [10] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [[hep-th/0212208](#)].
- [11] G. Bonelli, L. Bonora and F. Nesti, *Matrix string theory, 2D SYM instantons and affine Toda systems*, *Phys. Lett. B* **435** (1998) 303 [[hep-th/9805071](#)].
- [12] S.B. Giddings, F. Hacquebord and H.L. Verlinde, *High energy scattering and D -pair creation in matrix string theory*, *Nucl. Phys. B* **537** (1999) 260 [[hep-th/9804121](#)]; G. Bonelli, L. Bonora and F. Nesti, *String interactions from matrix string theory*, *Nucl. Phys. B* **538** (1999) 100 [[hep-th/9807232](#)]; G. Bonelli, L. Bonora, F. Nesti and A. Tomasiello, *Matrix string theory and its moduli space*, *Nucl. Phys. B* **554** (1999) 103 [[hep-th/9901093](#)]; *Heterotic matrix string theory and Riemann surfaces*, *Nucl. Phys. B* **564** (2000) 86 [[hep-th/9905092](#)].

- [13] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, *Comm. Number Theory Phys.* **1** (2007) 1 [[hep-th/0604151](#)];
S. Gukov and E. Witten, *Gauge theory, ramification and the geometric Langlands program*, [hep-th/0612073](#);
A. Kapustin, *Holomorphic reduction of $N = 2$ gauge theories, Wilson-'t Hooft operators and S -duality*, [hep-th/0612119](#);
E. Witten, *Gauge theory and wild ramification*, [arXiv:0710.0631](#).
- [14] N.J. Hitchin, *Stable bundles and integrable systems*, *Duke Math. Journ.* **54** (1987) 91.
- [15] R.S. Ward, *Integrable and solvable systems, and relations among them*, *Phil. Trans. Roy. Soc. Lond.* **A315** (1985) 451.
- [16] C.H. Taubes, *Arbitrary N : vortex solutions to the first order Landau-Ginzburg equations*, *Commun. Math. Phys.* **72** (1980) 277.
- [17] C. Rebbi, *Interaction of superconducting vortices*, in *Geometrical and topological methods in gauge theories: proceedings*, J.P. Harnad and S. Shnider eds., Lecture Notes in Physics volume 129, Springer-Verlag, Berlin Germany (1980).
- [18] M.K. Prasad and C.M. Sommerfield, *An exact classical solution for the 't Hooft monopole and the Julia-Zee dyon*, *Phys. Rev. Lett.* **35** (1975) 760.
- [19] N.Manton and P. Sutcliffe, *Topological solitons*, Cambridge University Press, Cambridge U.K. (2004).
- [20] M.A.C. Kneipp and D.I. Olive, *Solitons and vertex operators in twisted affine Toda field theories*, *Commun. Math. Phys.* **177** (1996) 561 [[hep-th/9404030](#)].
- [21] B. Konstant, *The principle three dimensional subgroup*, *Am. J. Math.* **81** (1959) 973.
- [22] N. Ganoulis, P. Goddard and D.I. Olive, *Selfdual monopoles and Toda molecules*, *Nucl. Phys.* **B 205** (1982) 601.
- [23] A.N. Leznov and M.V. Savelev, *Representation of zero curvature for the system of nonlinear partial differential equations $\chi(\alpha, z\bar{z}) = ((\kappa\chi)(\alpha))$ and its integrability*, *Lett. Math. Phys.* **3** (1979) 489; *Representation theory and integration of nonlinear spherically symmetric equations to gauge theories*, *Commun. Math. Phys.* **74** (1980) 111.
- [24] L. Bonora, C.P. Constantinidis, L.A. Ferreira and E.E. Leite, *Construction of exact Riemannian instanton solutions*, *J. Phys.* **A 36** (2003) 7193 [[hep-th/0208175](#)].
- [25] T.J. Hollowood, *Solitons in affine Toda field theories*, *Nucl. Phys.* **B 384** (1992) 523;
H. Aratyn, C.P. Constantinidis, L.A. Ferreira, J.F. Gomes and A.H. Zimerman, *Hirota's solitons in the affine and the conformal affine Toda models*, *Nucl. Phys.* **B 406** (1993) 727 [[hep-th/9212086](#)];
N.J. MacKay and W.A. McGhee, *Affine Toda solitons and automorphisms of Dynkin diagrams*, *Int. J. Mod. Phys.* **A 8** (1993) 2791 [*Erratum ibid.* **8** (1993) 3830] [[hep-th/9208057](#)];
D.I. Olive, N. Turok and J.W.R. Underwood, *Solitons and the energy momentum tensor for affine Toda theory*, *Nucl. Phys.* **B 401** (1993) 663; *Affine Toda solitons and vertex operators*, *Nucl. Phys.* **B 409** (1993) 509 [[hep-th/9305160](#)];
M.A.C. Kneipp and D.I. Olive, *Crossing and anti-solitons in affine Toda theories*, *Nucl. Phys.* **B 408** (1993) 565 [[hep-th/9305154](#)];
M.A.C. Kneipp, *Vertex operators, semiclassical limit for soliton S -matrices and the number of bound states in affine Toda field theories*, *Nucl. Phys.* **B 577** (2000) 390 [[hep-th/9909128](#)].